Influence of a rough thin layer on the potential

Ionel Sorin Ciuperca, Ronan Perrussel, Clair Poignard

Influence of a rough thin layer on the potential

I. Ciuperca*, R. Perrussel†, and C. Poignard‡;
*Université de Lyon, ICJ UMR CNRS 5208, Lyon, F-69003;
†Laboratoire Ampère UMR CNRS 5005, Université de Lyon, École Centrale de Lyon, F-69134 Écully, France
‡ INRIA Bordeaux-Sud-Ouest, Team MC2, IMB UMR CNRS 5251, Université Bordeaux1, Talence 33405, France
mail to: clair.poignard@inria.fr

***Abstract*—In this paper, we study the behavior of the steady-state voltage potentials in a material composed by an interior medium surrounded by a rough thin layer and embedded in an ambient bounded medium. The roughness of the layer is supposed to be ε–periodic, ε being the small thickness of the layer. We present and validate numerically the rigorous approximate transmissions proved by Ciuperca *et al.* in [1]. This paper extends previous works in which the layer had a constant thickness.**

I. Introduction

In the domains with a rough thin layer, numerical difficulties appear due to the complex geometry of the rough layer when computing the steady-state potentials. We present here how these difficulties may be avoided by replacing this rough layer by appropriate transmission conditions. Particularly, we show that considering only the mean effect of the roughness is not sufficient to obtain the potential with a good accuracy.

A. Statement of the problem

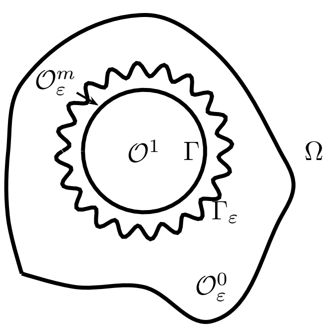


Fig. 1. Geometry of the problem.

Let Ω be a smooth bounded domain of $\mathbb{R}^2$ with connected boundary $\partial\Omega$. For $\varepsilon > 0$, we split Ω into three subdomains: $\mathcal{O}^1$, $\mathcal{O}^m_\varepsilon$ and $\mathcal{O}^0_\varepsilon$. $\mathcal{O}^1$ is a smooth domain strictly embedded in Ω (see Fig. 1). We denote by Γ its connected boundary. The domain $\mathcal{O}^m_\varepsilon$ is a thin oscillating layer surrounding $\mathcal{O}_1$. We denote by Γ_ε the oscillating boundary of $\mathcal{O}^m_\varepsilon$: $\Gamma_\varepsilon = \partial\mathcal{O}^m_\varepsilon \setminus \Gamma$. The domain $\mathcal{O}^0_\varepsilon$ is defined by: $\mathcal{O}^0_\varepsilon = \Omega \setminus \overline{(\mathcal{O}^1 \cup \mathcal{O}^m_\varepsilon)}$. We also denote by $\mathcal{O}^0 = \Omega\setminus\overline{\mathcal{O}^1}$. Two piecewise-constant conductivities on the domain Ω have to be defined:

$$\sigma(z) = \begin{cases} \sigma_1, \text{ if } z \in \mathcal{O}^1, \\ \sigma_m, \text{ if } z \in \mathcal{O}^m_\varepsilon, \\ \sigma_0, \text{ if } z \in \mathcal{O}^0_\varepsilon. \end{cases} \qquad \tilde\sigma(z) = \begin{cases} \sigma_1, \text{ if } z \in \mathcal{O}^1, \\ \sigma_0, \text{ if } z \in \Omega \setminus \mathcal{O}^1. \end{cases}$$

where σ_1, σ_m and σ_0 are given positive constants[1].

Let u^ε and u^0 be defined by:

$$\begin{cases} \nabla.\left(\sigma\nabla u^\varepsilon\right) = 0, \text{ in } \Omega, \\ u^\varepsilon|_{\partial\Omega} = g, \end{cases} , \begin{cases} \nabla.\left(\tilde\sigma\nabla u^0\right) = 0, \text{ in } \Omega, \\ u^0|_{\partial\Omega} = g, \end{cases} \tag{1}$$

where g is a sufficiently smooth boundary data. We present how to define the potential u^1 such that u^ε is approached by $u^\varepsilon = u^0 + \varepsilon u^1 + o(\varepsilon^{3/2})$ for ε tending to zero[2].

II. Heuristics of the derivation of the conditions

Suppose Γ is a smooth closed curve of $\mathbb{R}^2$ of length 1 and parameterize it by the curvilinear coordinate $\Gamma = \{\Psi(\theta), \theta \in [0,1]\}$. Let n be the (outward) normal to $\partial\mathcal{O}_1$. Γ_ε is described by

$$\Gamma_\varepsilon = \{\Psi(\theta) + \varepsilon f(\theta/\varepsilon)n(\theta), \theta \in [0,1]\},$$

where f is a smooth 1–periodic and positive function, which describes the roughness of the layer.

A. Boundary layer corrector in the infinite strip

The key-point of the derivation of the equivalent transmission conditions consists in taking advantage of the periodicity of the roughness. This is performed by unfolding and upscaling the rough thin layer into the infinite strip $\mathbb{R} \times [0,1]$.

Define the closed curves $\mathcal{C}_1$ and $\mathcal{C}_0$, which are trigonometrically oriented by

$$\mathcal{C}_0 = \{0\} \times [0,1], \quad \mathcal{C}_1 = \{(f(y), y), \forall y \in [0,1]\}.$$

The outward normals to $\mathcal{C}_0$ and $\mathcal{C}_1$ equal

$$n_{\mathcal{C}_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n_{\mathcal{C}_1} = \frac{1}{\sqrt{1 + (f'(y))^2}} \begin{pmatrix} 1 \\ -f'(y) \end{pmatrix}. \tag{2}$$

According to [1] there exists a unique couple (A^0, a^0) where A^0 is a continuous vector field and a^0 is constant such that

$$A^0 \text{ is 1-periodic in } y, \quad \Delta A^0 = 0, \text{ in } \mathbb{R} \times [0,1], \tag{3a}$$

$$\sigma_0\partial_n A^0|_{\mathcal{C}_1^+} - \sigma_m\partial_n A^0|_{\mathcal{C}_1^-} = (\sigma_m - \sigma_0)n_{\mathcal{C}_1}, \tag{3b}$$

$$\sigma_m\partial_n A^0|_{\mathcal{C}_0^+} - \sigma_1\partial_n A^0|_{\mathcal{C}_0^-} = -(\sigma_m - \sigma_0)n_{\mathcal{C}_0}, \tag{3c}$$

$$A^0 \to_{x\to-\infty} 0, \; A^0 - a^0 \to_{x\to+\infty} 0, \tag{3d}$$

where the convergences at infinity are exponential. We emphasize that a^0 is not imposed but is a floating potential.

[1]The same following results hold if σ_0, σ_1, and σ_m are given complex numbers with imaginary parts (and respectively real parts) with the same sign.

[2]The notation $o(\varepsilon^{3/2})$ means that $\left\|u^\varepsilon - (u^0 + \varepsilon u^1)\right\|$ goes to zero faster than $\varepsilon^{3/2}$ as ε goes to zero. We refer to Theorem 1.1 of [1] for a precise description of the involved norms and the accuracy of the convergence.

B. Approximate transmission conditions

Our transmission conditions are then obtained with the help of the constant vectors D_1 and D_2 defined by:

$$
\begin{aligned}
D_1 &= (\sigma_0 - \sigma_m)\left[\int_0^1 f(y)\,dy\, n_{\mathcal{C}_0} + \int_0^1 A^0(f(y),y)\,dy\right] \\
&\quad + (\sigma_m - \sigma_1)\int_0^1 A^0(0,y)\,dy - \sigma_0 a^0, \\
D_2 &= (\sigma_m - \sigma_0)\left[\int_0^1 A^0(f(y),y) f'(y)\,dy \right.\\
&\quad \left. - \int_0^1 f(y)\,dy \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].
\end{aligned}
$$

The potential u^1 is then defined by[3]:

$$
\begin{cases}
\Delta u^1 = 0, \text{ in } \mathcal{O}^0 \cup \mathcal{O}^1, \quad u^1|_{\partial\Omega} = 0, \\
\left[\tilde{\sigma}\partial_n u^1\right]_\Gamma = -\kappa D_1 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma+} \end{pmatrix} + D_2 \cdot \partial_t \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma+} \end{pmatrix}, \\
\left[u^1\right]_\Gamma = a^0 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma+} \end{pmatrix},
\end{cases}
$$

where ∂_t and ∂_n denote the tangential and the normal derivatives along Γ and κ is the curvature of Γ.

We emphasize that our conditions are different than if we would only consider the mean effect of the roughness. In this case, denoting by $\overline{f}$ the mean of f, the conditions would be (see [1], [3], [4]):

$$
\left[\tilde{\sigma}\partial_n \tilde{u}^1\right]_\Gamma = (\sigma_0 - \sigma_m)\overline{f}\partial_t^2 u^0|_\Gamma, \; \left[\tilde{u}^1\right]_\Gamma = \frac{\sigma_0 - \sigma_m}{\sigma_m}\overline{f}\partial_n u^0|_{\Gamma+}.
$$

III. Numerical simulations

In order to verify the convergence rate stated in Section I, we consider a problem where the geometry and the boundary conditions are $\varepsilon-$periodic. The computational domain Ω is delimited by the circles of radius 2 and of radius 0.2 centered in 0, while $\mathcal{O}^1$ is the intersection of Ω with the concentric disk of radius 1. The rough layer is then described by $f(y) = 1 + 1/2\sin(y)$. One period of the domain is shown Fig 2(a). The Dirichlet boundary data is identically 1 on the outer circle and 0 on the inner circle.

The mesh generator *Gmsh* [2] and the finite element library *Getfem++* [5] enables us to compute the four potentials u^ε, u^0, u^1 and $\tilde{u}^1$.

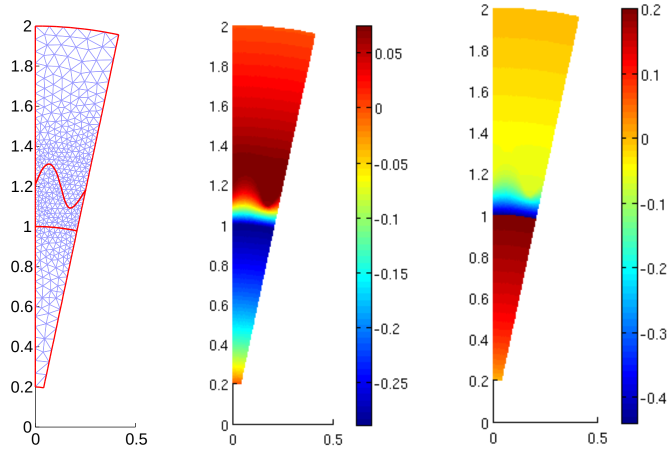
The rough thin layer is supposed slighty insulating. The conductivities σ_0, σ_1 and σ_m respectively equal to 3, 1 and 0.1. The computed coefficients[4] issued from Problem (3) are given in Table I.

TABLE I
COEFFICIENTS ISSUED FROM THE SOLUTION TO PROBLEM (3). 3 SIGNIFICANT DIGITS ARE KEPT.

a_1^0	a_2^0	D_1^1	D_2^1	D_1^2	D_2^2
19.3	0	0	0	-0.0499	-3.87

[3]We denote by $[w]_\Gamma$ the jump of a function w on Γ.

[4]The convergences at the infinity in Problem (3) are exponential hence we just have to compute problem (3) for $|x| \leq M$, with M large enough to obtain a^0 with a good accuracy.



(a) One period. (b) Error order 0. (c) Error order 1.

Fig. 2. Representation of one period of the domain and the corresponding errors with approximate solutions u^0 and $u^0 + \epsilon u^1$. $\epsilon = 2\pi/30$. Do not consider the error in the rough layer because a proper reconstruction of the solution in it is not currently implemented.

The numerical convergence rates for the H^1-norm in $\mathcal{O}^1$ of the three following errors $u^\varepsilon - u^0$, $u^\varepsilon - u^0 - \varepsilon u^1$ and $u^\varepsilon - u^0 - \varepsilon\tilde{u}^1$ as ε goes to zero are given Figure 2. As predicted by the theory, the rates are close to 1 for the order 0 and for the order 1 with the mean effect, whereas it is close to 2 for the "real" order 1 equal to $u^\varepsilon - u^0 - \varepsilon u^1$.

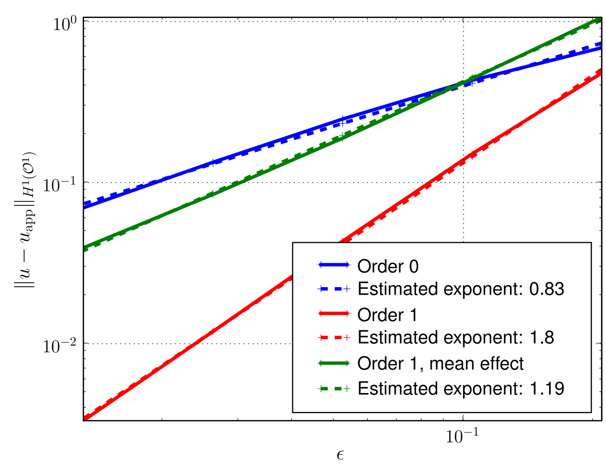


Fig. 3. H^1-Error in the cytoplasm vs ϵ for three approximate solutions.

References

[1] S. Ciuperca, M. Jai, and C. Poignard. Approximate transmission conditions through a rough thin layer. The case of the periodic roughness. Research report INRIA RR-6812. http://hal.inria.fr/inria-00356124/fr/.

[2] C. Geuzaine and J. F. Remacle. Gmsh mesh generator http://geuz.org/gmsh.

[3] C. Poignard. Approximate transmission conditions through a weakly oscillating thin layer. *Math. Meth. App. Sci.*, 32:435–453, 2009.

[4] C. Poignard, P. Dular, R. Perrussel, L. Krähenbühl, L. Nicolas, and M. Schatzman. Approximate conditions replacing thin layer. *IEEE Trans. on Mag.*, 44(6):1154–1157, 2008.

[5] Y. Renard and J. Pommier. Getfem finite element library. http://home.gna.org/getfem/.